Conjugacy of involutive antiautomorphisms of von Neumann algebras

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1. Introduction

Let M be a von Neumann algebra and α a central involution of M, i.e. α is a $*$-antiautomorphism of order 2 leaving the center of M elementwise fixed. Then the set $M^\alpha = \{x \in M: \ x=x^* = \alpha(x)\}$ is a JW-algebra with Jordan product $x \circ y = \frac{1}{2}(xy+yx)$. In this paper we shall study the relationship between M^α and M^β for two central involutions α and β. The main result states that α and β are (centrally) conjugate, i.e. there is a $*$-automorphism ϕ of M leaving the center elementwise fixed such that $\beta = \phi\alpha\phi^{-1}$ if and only if M^α and M^β are isomorphic as Jordan algebras via an isomorphism which leaves the center elementwise fixed. Now M^α generates M as a von Neumann algebra (except in a few simple cases) and there are von Neumann algebra with many conjugacy classes of central involutions. Thus there may be many, even an uncountable number, of nonisomorphic JW-algebras which generate the same von Neumann algebra. Such examples are exhibited in section 5.

The main result indicates that it is not so easy for two central involutions to be conjugate. This problem is taken up in section 4, where it is shown that if two central involutions α and β are in a sense close then they tend to be conjugate. For example if $\alpha\beta$ is an inner automorphism then α and β are conjugate "modulo" the two nonconjugate involutions of the complex

2×2 matrices. If $\|\alpha-\beta\| < 2$ then they are necessarily conjugate.

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2. THE TYPE I CASE

Definition 2.1. By an involution of a von Neumann algebra M we shall mean a $*$-antiautomorphism of order 2. If α is a $*$-automorphism or $*$-antiautomorphism of M we say α is central if α leaves the center of M elementwise fixed. Two central involutions α and β are said to be centrally conjugate, written $\alpha \sim \beta$, if there is a central $*$-automorphism γ of M such that $\alpha = \gamma\beta\gamma^{-1}$.

Recall that a JW-algebra is a weakly closed Jordan algebra of selfadjoint operators on a complex Hilbert space with the Jordan product $a \circ b = \frac{1}{2}(ab+ba)$. We shall refer to [8] for the theory of JW-algebras. If α is an involution of a von Neumann algebra M we denote by M^α the set $\{x\in M;\ x=x^*,\ \alpha(x)=x\}$, i.e. M^α is the fixed point set of α in M_{sa}. We let $R^\alpha = \{x\in M\colon \alpha(x)=x^*\}$. Then R^α is a weakly closed real $*$-algebra such that M is the direct sum $M = R^\alpha + iR^\alpha$, and $\alpha(x+iy) = x^* + iy^*$, $x, y \in R^\alpha$, see [8, 7.3.2]. Furthermore $M^\alpha = R^\alpha_{sa}$, hence M^α is a reversible JW-algebra, viz $x_1,\dots,x_n \in M$ implies $x_1x_2\cdots x_n + x_nx_{n-1}\cdots x_1 \in M$.

LEMMA 2.2. Let M be a von Neumann algebra with central involutions α and β. Suppose $\alpha\beta = \mathrm{Ad}\,u$ for a unitary u in M. Then there is a symmetry s in the center of M such that

$$\alpha(u) = \beta(u) = su$$

Proof. Since $\alpha\beta(u) = \mathrm{Ad}u(u) = u$, $\alpha(u) = \beta(u)$, whence $\beta(u) = \alpha(u) = \mathrm{Ad}u(\beta(u)) = u\beta(u)u^*$. Thus u commutes with both $\alpha(u)$ and $\beta(u)$. Let $x \in R^\beta$. Since $\iota = \alpha^2 = (\mathrm{Ad}u \circ \beta)^2$, where ι is the identity map, $\beta = \mathrm{Ad}u\,\beta\,\mathrm{Ad}u$, and so

$$x^* = \beta(x) = u(\beta(uxu^*))u^* = u\beta(u^*)x^*\beta(u)u^*.$$

Since $M = R^\beta + iR^\beta$ it follows that $u\beta(u)^*$ belongs to the center Z of M, whence $\beta(u) = su$ for a unitary $s \in Z$. Since $u = \beta(\beta(u)) = \beta(su) = s\beta(u) = s^2u$, $s^2 = 1$, and so s is a symmetry.

LEMMA 2.3. Let M be a von Neumann algebra with center Z. Suppose α is a central involution such that M^α is abelian. Then $M^\alpha = Z_{sa}$, and M is the direct sum $M = M_1 \oplus M_2$, where M_i is a von Neumann algebra of type I_i.

Proof. Let $x \in M_{sa}$, $a \in M^\alpha$. Since $\alpha(x) + x \in M^\alpha$ and M^α is abelian, $[a, \alpha(x)+x] = 0$, hence $[a, \alpha(x)] = [x,a]$. Thus $\alpha([a,x]) = [\alpha(x), a] = [a,x]$, so $i[a,x] \in M^\alpha$ for all self-adjoint $x \in M$.

Suppose $M^\alpha \neq Z$. Let e be a projection in M^α such that $e \notin Z$. By the Comparison theorem there is a projection $q \in Z$ such that $eq \prec (1-e)q$ and $(1-e)(1-q) \prec e(1-q)$. Since $e \notin Z$ not both eq and $(1-e)(1-q)$ can be zero. If we cut M down by q or $1-q$ and replace e by $1-e$ if necessary we may assume $e \prec 1-e$. Thus there exists a symmetry $s \in M$ such that $ses = f < 1-e$. By the first paragraph of the proof $i(es-se) \in M^\alpha$, so it commutes with e. Thus

$$ese - se = (es-se)e = e(es-se) = es - ese.$$

But $s(ese) = fe = 0$, so $ese = 0$. Thus $-se = es$, hence multiplication from the right by s yields $-f = e$, a contradiction. Thus $M^{\alpha} = Z$, so by [8, 7.3.8] $M = M_1 \oplus M_2$ with M_i of type I_i. Q.E.D.

Recall that a von Neumann algebra is homogeneous of type I_n, $n \in \mathbb{N} \cup \{\infty\}$, if there is a factor M_n of type I_n such that $M \cong M_n \otimes Z$, where Z is abelian. Z is identified with the center of M and M_n with $B(H_n)$, where H_n is a Hilbert space of dimension n if $n \in \mathbb{N}$, and infinite if $n = \infty$. If $n \in \mathbb{N}$ we denote by t_n the transpose map of $B(H_n)$ with respect to some orthonormal basis, and if $n = \infty$ we let t_{∞} denote the infinite dimensional version of t_n, called the real flip in [8]. Note that a type I von Neumann algebra is a direct sum of homogeneous ones. Hence it is sufficient to study central involutions of homogeneous algebras.

LEMMA 2.4. Let M be a von Neumann algebra which is homogeneous of type I_n. Suppose α is a central involution on M and p a projection in M^{α} which is abelian in M and has central support 1. Then M and α can be written in the form $M = B(H_n) \otimes Z$, $\alpha = t_n \otimes \iota$.

Proof. Since p is abelian and has central support 1, $M_p = Z_p \cong Z$, where Z is the center of M. By [8, 5.3.3] M^{α} is a JW-algebra of type I_n and so contains n mutually orthogonal strongly connected abelian projections $p_1 = p, p_2, \ldots, p_n$ (or an infinite number if $n=\infty$) with sum 1 and partial symmetries s_{ij} exchanging p_i and p_j such that $s_{ij}^2 = p_i + p_j$, $i \neq j$, and such that $e_{ij} = p_i s_{ij} p_j$ form a complete set of matrix units in M, see [8, 5.3 and proof of 7.6.3]. Thus $\alpha(e_{ij}) = \alpha(p_i s_{ij} p_j) = e_{ji}$.

Since the weakly closed span of the e_{ij} is $B(H_n)$ and $\alpha|B(H_n) = t_n$, $M \cong B(H_n) \otimes Z$ via an isomorphism which carries α onto $t_n \otimes \iota$. Q.E.D.

LEMMA 2.5. Let M be a von Neumann algebra which is homogeneous of type I_2. Suppose α is a central involution of M such that $M^\alpha = Z$, where Z is the center of M. Then M is of the form $M = B(H_2) \otimes Z$ and α of the form $q \otimes \iota$, where q is the involution of $B(H_2)$ given by

$$q(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}).$$

Proof. We may write M in the form $M = B(H_2) \otimes Z$. Then $\beta = t_2 \otimes \iota$ is another central involution of M, and $\alpha\beta$ is a central automorphism of M, hence of the form $\alpha\beta = \mathrm{Ad}u$. By Lemma 2.2 $\alpha(u) = su$ for a symmetry $s \in Z$, $s = e - f$ with e, f central projections with sum 1. Then $\alpha(eu) = eu \in M^\alpha = Z$, so $\alpha|Me = \beta|Me$, a case we have excluded since M^β is of type I_2. We thus have $\alpha(u) = -u$. Since $\alpha(u^2) = u^2$, $u^2 \in Z$. Let z be a square root of u^2 in Z, so z is unitary and $z^2 = u^2$. Let $v = z^{-1}u$. Then v is a symmetry such that $\alpha(v) = -v$, and so $v = g - h$ for two projections with sum 1 satisfying $\alpha(g) = h$. But then $g \sim h$ as projections in M, [12, Lem. 3.3], so there is a symmetry $t \in M$ such that $tgt = h$, and therefore $vtv = -t$. Since $t + \alpha(t) = w \in Z$, $\alpha(t) = w - t$. Thus $1 = \alpha(t)^2 = w^2 - 2wt + 1$, so that $w(w-2t) = 0$. Let r be a central projection such that $wr = 0$, $w(1-r)$ is nonsingular. In the former case $\alpha(rt) = -rt$; in the latter $2t(1-r) = w(1-r) \in Z$, which is impossible since $tgt = h$. Therefore $w = 0$, and $\alpha(t) = -t$. Let $e_{11} = g$, $e_{22} = h$, $e_{12} = gth$, $e_{21} = htg$. Then e_{ij} form a complete set of 2×2 matrix units.

Hence they span $B(H_2)$ and $\alpha \mid B(H_2) = q$. The rest is clear.

Q.E.D.

THEOREM 2.6. Let M be von Neumann algebra which is homogeneous of type I_n, $n \in \mathbb{N} \cup \{\infty\}$. Suppose α is a central involution on M. Then M is a direct sum $M = M_1 \oplus M_2$ of two von Neumann algebra and M_1 and M_2 such that:

$M_1 = B(H_n) \otimes Z$ with Z abelian, and $\alpha|M_1 = t_n \otimes \iota$,

$M_2 = B(H_m) \otimes B(H_2) \otimes Z$ with Z abelian, $2m = n$, and $\alpha|M_2 = t_m \otimes q \otimes \iota$

Proof. We first assume $n < \infty$. Since every von Neumann subalgebra of M is of type I, so is M^{α} by [8, 7.4.3]. Let p be an abelian projection in M^{α} with central support 1. By Lemma 2.3 $M_p = N_1 \oplus N_2$ with N_i of type I_i. We consider the two cases separately. If M_p is of type I_1 the theorem follows from Lemma 2.4. If M_p is of type I_2 then n is even. Let $m = \frac{n}{2}$. As in the proof of Lemma 2.4 we can find matrix units e_{ij}, $1 \leq i,j \leq m$, such that $\alpha(e_{ij}) = e_{ji}$. We thus have $M = B(H_m) \otimes N$, where N is of type I_2, and $\alpha = t_m \otimes \alpha|N$. An application of Lemma 2.5 now completes the proof when $n < \infty$.

Finally assume $n = \infty$. Let p be an abelian projection in M with central support 1. Then $q = p \vee \alpha(p) \in M^{\alpha}$, and M_q is of type I_1 or I_2. In particular $(M^{\alpha})_q$ is of type I_1 or I_2. We may thus complete the proof as in the preceding paragraph.

Q.E.D.

COROLLARY 2.7. Let M be a von Neumann algebra of type I, and let α and β be central involutions on M. Then we have

(i) $\alpha \sim \beta$ if and only if $M^{\alpha} \cong M^{\beta}$ via an isomorphism which leaves the center of M elementwise fixed.

(ii) There are central projections e and f in M with sum 1 such that $\alpha\,|Me \sim \beta|\,Me$, and $(\alpha|Mf) \otimes t_2 \sim (\beta|Mf) \otimes q$ as involutions on $(Mf) \otimes B(H_2)$.

The proof is an easy case by case check using Theorem 2.6, and is omitted. The reader should just keep in mind that $t_2 \otimes t_2 = t_4 = q \otimes q$, and $t_n \otimes t_2 = t_{n+2}$ is never conjugate to $t_n \otimes q$.

3. CONJUGACY AND JORDAN ALGEBRAS.

Let A be a JW-algebra. By [8, 7.1.9 and 7.2.8), or by [7], there exist up to isomorphism a unique von Neumann algebra $W^*(A)$ and a normal isomorphism $\psi\colon A \to W^*(A)$ with the following properties:

(i) $\psi(A)$ generates $W^*(A)$ as a von Neumann algebra.

(ii) If B is a von Neumann algebra and $\phi\colon A \to B_{sa}$ is a normal homomorphism (i.e. ϕ is linear and preserves the Jordan product) then there is a normal $*$-homomorphism $\hat{\phi}\colon W^*(A) \to B$ such that $\hat{\phi} \circ \psi = \phi$.

(iii) There is an involution Φ on $W^*(A)$ such that $\Phi(\psi(a)) = \psi(a)$ for all $a \in A$.

$W^*(A)$ is called the <u>universal von Neumann algebra</u> of A and Φ the <u>canonical antiautomorphism</u> of $W^*(A)$.

<u>LEMMA 3.1</u>. Let M be a von Neumann algebra and α a central involution of M such that M^α generates M as a von Neumann algebra, and such that M^α has no parts of type I_1 and I_2.

Then there is an isomorphism $\gamma: M \to W^*(M^\alpha)$ such that $\alpha = \gamma^{-1}\Phi\gamma$, and $\gamma(x) = \psi(x)$ for all self-adjoint x in the center of M.

Proof. Note that $eM^\alpha \neq eM_{sa}$ for all nonzero central projections e in M. Indeed, if $eM^\alpha = eM_{sa}$ let $x,y \in eM^\alpha$. Then $xy \in eM^\alpha + ie\,M^\alpha$ and so $xy = \alpha(xy) = yx$, proving that eM^α is of type I_1, contrary to assumption. It then follows from [8, 7.3.5] that, since M^α has no portion or type I_1, the canonical antiautomorphism Φ leaves the center of $W^*(M^\alpha)$ pointwise invariant, i.e. Φ is a central involution. If ψ is the imbedding of M^α in $W^*(M^\alpha)$ then by [8, 7.3.3] $\psi(M^\alpha) = W^*(M^\alpha)^\Phi$; in particular the center of $\psi(M^\alpha)$ contains that of $W^*(M^\alpha)_{sa}$. Since $\psi(M^\alpha)$ generates $W^*(M^\alpha)$ the converse inclusion is trivial, so the two centers coincide.

Let by property (ii) in the definition of $W^*(M^\alpha)$, ϕ be the normal $*$-homomorphism $\phi: W^*(M^\alpha) \to M$ such that $\phi\psi(x) = x$ for $x \in M^\alpha$. Then ϕ is an isomorphism. Indeed, if e is a central projection in $W^*(M^\alpha)$ such that $\phi(e) = 0$, then by the previous paragraph $e = \psi(f)$ with f a central projection in M^α, hence $f = \phi\psi(f) = 0$, so $e = 0$. Since by assumption M^α generates M, ϕ is surjective, proving the assertion. Let $\gamma = \phi^{-1}$. Then γ is an isomorphism of M onto $W^*(M^\alpha)$ such that if x is in the center of M then $\gamma(x) = \psi(x)$.

Let R be the weakly closed real $*$-algebra generated by M^α. If $x_1,\dots,x_n \in M^\alpha$ and $x = x_1x_2\dots x_n$, then $\alpha(x) = x_nx_{n-1}\dots x_1 = x^*$, hence $R \subset R^\alpha = \{x \in M: \alpha(x) = x^*\}$. Since $R^\alpha \cap iR^\alpha = \{0\}$ the same is true for R. Thus $R + iR$ is von Neumann algebra [13], hence equal to M by assumption. If $z = x + iy \in R^\alpha$ with $x, y \in R$, then $x^* - iy^* = z^* = \alpha(z) = x^* + iy^*$, so that $y = 0$, and $z \in R$. Therefore $R = R^\alpha$. By construction of $W^*(M^\alpha)$ and Φ

we know that $R^{\Phi} = \{x \in W^{*}(M^{\alpha}) : \phi(x) = x^{*}\}$ is the weakly closed real $*$-algebra generated by $\psi(M^{\alpha})$, and so $\phi(R^{\Phi}) = R^{\alpha}$. Thus if $x,y \in R^{\alpha}$ we have

$$\gamma\alpha(x+iy) = \gamma(x^{*}+iy^{*}) = \phi^{-1}(x)^{*} + i\phi^{-1}(y)^{*}$$
$$= \Phi(\phi^{-1}(x+iy)) = \phi\gamma(x+iy).$$

Q.E.D.

PROPOSITION 3.2. Let M be a von Neumann algebra with no type I portion. Suppose α is a central involution on M. Then M^{α} generates M as a von Neumann algebra, and there exists an isomorphism $\gamma : M \to W^{*}(M^{\alpha})$ such that $\alpha = \gamma^{-1}\Phi\gamma$, and $\gamma(x) = \psi(x)$ for all self-adjoint x in the center of M.

Proof: Note that M^{α} has no type I portion. Indeed, if there is an abelian projection p in M^{α} then α restricts to a central involution of M_p. Since the fixed points of $\alpha|(M_p)_{sa}$ is M_p^{α}, it follows from Lemma 2.3 that M_p is of type I, contradicting the assumption that M has not type I portion.

Let R be the weakly closed real $*$-algebra generated by M^{α}, and let $N = R + iR$. As in the proof of Lemma 3.1 $R \subset R^{\alpha}$ and N is a von Neumann algebra, which by Lemma 3.1 is isomorphic to $W^{*}(M^{\alpha})$. Furthermore, by [8, 7.4.2] N has no type I portion. But then by [12, Lem. 2.12] and its proof there are projections e, f in M^{α} with sum 1 and a symmetry $s \in M^{\alpha}$ such that $ses = f$, and such that the unitary $u = (e-f)s$ satisfies $u^{*}su = -s$ and $u^{*} = -u$.

Let $M_{\pm} = \{x \in M : \alpha(x) = \pm x\}$. Then M is the direct sum $M = M_{+} \oplus M_{-}$. Suppose $x,y \in R^{\alpha}$ and $x + iy \in M_{+}$. Then $x + iy = \alpha(x+iy) = x^{*} + iy^{*}$, hence both $x,y \in M^{\alpha} = (R^{\alpha})_{sa}$. Thus $M_{+} =$

$M^\alpha + iM^\alpha$. In particular $M_+ \subset N$. Let u be as in the previous paragraph. Then $\alpha(u) = u^* = -u$, so $u \in R \cap M_-$. If $x \in M_-$ then clearly $ux + xu \in M_+ \subset N$. Therefore, since $u \in N$, $x + uxu^* \in N$.

Let $\rho = \mathrm{Ad}u$. Then $\rho^2 = \iota$, and $P = \frac{1}{2}(\iota+\rho): M \to N$. Since $P(M) = M^\rho = \{x \in M: \rho(x) = x\}$, $M^\rho \subset N$. Now the symmetry $s \in M^\alpha$ used in the construction of u belongs to $M_-^\rho = \{x \in M: \rho(x) = -x\}$, hence if $x \in M_-^\rho$ then $xs \in M^\rho \subset N$. Thus $x = (xs)s \in N$. In particular $M = M^\rho + M_-^\rho \subset N$, and they are equal. An application of Lemma 3.1 completes the proof.

The next result extends part (i) of Corollary 2.7 to the non-type I case.

THEOREM 3.3. Let M be a von Neumann algebra with center Z. Suppose α and β are central involutions on M. Then $\alpha \sim \beta$ if and only if M^α and M^β are isomorphic as JW-algebras via an isomorphism which leaves Z_{sa} elementwise fixed.

Proof. By Corollary 2.7 we may assume M has no type I portion. If γ is a central automorphism of M such that $\alpha = \gamma\beta\gamma^{-1}$ then clearly γ is an isomorphism of M^β onto M^α leaving Z_{sa} elementwise fixed.

Conversely suppose there is an isomorphism of M^α onto M^β leaving Z_{sa} elementwise fixed. By uniqueness of the universal von Neumann algebra there exists an isomorphism $\theta: W^*(M^\alpha) \to W^*(M^\beta)$ carrying M^α onto M^β and leaving Z_{sa} elementwise fixed. By Proposition 3.2 there are isomorphisms γ_α and γ_β of M onto $W^*(M^\alpha)$ and $W^*(M^\beta)$ respectively such that $\gamma_\alpha(x) = \phi_\alpha(x)$ and $\gamma_\beta(x) = \phi_\beta(x)$ for $x \in Z_{sa}$, where ϕ_α is the imbedding of M^α in $W^*(M^\alpha)$, and similarly for ϕ_β. Furthermore, if

Φ_α and Φ_β are the canonical antiautomorphisms of $W^*(M^\alpha)$ and $W^*(M^\beta)$ then $\alpha = \gamma_\alpha^{-1}\Phi_\alpha\gamma_\alpha$ and $\beta = \gamma_\beta^{-1}\Phi_\beta\gamma_\beta$. Now by construction of the universal algebra and the canonical antiautomorphism, $\Phi_\beta = \theta\Phi_\alpha\theta^{-1}$. Thus we have

$$\beta = \gamma_\beta^{-1}\Phi_\beta\gamma_\beta = \gamma_\beta^{-1}\theta\Phi_\alpha\theta^{-1}\gamma_\beta = \gamma_\beta^{-1}\theta\gamma_\alpha\alpha\gamma_\alpha^{-1}\theta^{-1}\gamma_\beta,$$

so that $\beta = \gamma^{-1}\alpha\gamma$, where γ is the central automorphism $\gamma_\alpha^{-1}\theta^{-1}\gamma_\beta$ of M. Q.E.D.

In the factor case Theorem 3.3 has a very simpel form.

COROLLARY 3.4. Let M be a factor with involutions α and β. Then α and β are conjugate if and only if $M^\alpha \cong M^\beta$.

If α is a central involution of the von Neumann algebra M and e is a projection in M^α then $\alpha|M_e$ is a central involution, and $(M_e)^\alpha = (M^\alpha)_e$. Theorem 3.3 has a natural application to conjugacy of restrictions like $\alpha|M_e$.

THEOREM 3.5. Let M be a von Neumann algebra with central involutions α and β. Suppose there is a projection $e \in M^\alpha \cap M^\beta$ with central support 1 such that $\alpha|M_e \sim \beta|M_e$. Then $\alpha \sim \beta$.

Proof. We first assume M is of type I. By Lemma 2.3 if p is an abelian projection in $(M^\alpha)_e$ then M_p is a direct sum of two von Neumann algebras of types I_1 and I_2 respectively. Since $\alpha|M_e \sim \beta|M_e$ the same must be true for an abelian projection in $(M^\beta)_e$. Thus by Theorem 2.6 we have $\alpha \sim \beta$.

Next consider the case when M is of type II_1. Let Z

denote the center of M and let $\Phi: M \to Z$ be the center valued trace, and let p be a central projection such that $\Phi(e)p \geq \frac{1}{n}p$. By repeated use of the Halving Lemma [8, 5.2.14] we can find an orthogonal family $e_1,\dots,e_{2^k}$ of equivalent projections in $M^\alpha p$ such that $p = e_1+\dots+e_{2^k}$, and $2^{-k} \leq \frac{1}{n}$. Then $\Phi(e_i) = 2^{-k}p$ for all i. An application of the Comparison theorem [8, 5.2.13] shows that ep e_i for all i, so that ep contains a subprojection f with $\Phi(f) = 2^{-k}p$. Since it suffices to show the restrictions $\alpha|M_p$ and $\beta|M_p$ are centrally conjugate for each central projection p, we may assume that e has central support 1, and $f \in M^\alpha$ is a projection majorized by e such that f can be extended to an orthogonal family of m equivalent projections in M^α with sum 1. By the Halving Lemma [8, 5.2.14] there are two equivalent projections p_1 and p_2 in M^α with sum f. We therefore have $2m$ equivalent projections $p_1,\dots,p_{2m}$ in M^α with sum 1. Since orthogonal equivalent projections are strongly connected [8, 5.2.8] the Coordinatization theorem for special Jordan algebras [8, 2.8.3] shows $M^\alpha = H_{2m}(R_\alpha)$ - the hermitian $2m \times 2m$ matrices over a $*$-algebra R_α. Then $M_f \cong H_2(R_\alpha)$. From the proof of [8,2.8.3] we have that if ${}_0 = \{a \in M_{2m}(R_\alpha) : ae_{ij} = e_{ij}a$ for all $i,\}$, where (e_{ij}) is the given complete set of matrix units in $M_{2m}(R_\alpha)$, then $R_\alpha = \{a \in {}_0 : ae_{12} + a^* e_{21} \in H_{2m}(R_\alpha)\}$. In particular $Z_{sa} = \{a1 : a$ is self-adjoint in the center of $R_\alpha\}$.

Let now $\phi: M_e^\alpha \to M_e^\beta$ be the given central isomorphism. Then $\phi(f) = \phi(p_1) + \phi(p_2)$, and we may as for α show $M^\beta \cong H_{2m}(R_\beta)$ and $M^\beta_{\phi(f)} \cong H_2(R_\beta)$, where R_β is a $*$-algebra. Since ϕ restricts to an isomorphism $M_f^\alpha \to M^\beta_{\phi(f)}$, $H_2(R_\alpha) \cong H_2(R_\beta)$, hence by the above paragraph $M^\alpha \cong H_{2m}(R_\alpha) \cong H_{2m}(R_\beta) \cong M^\beta$. Since $\phi|Z_{sa}e$ is the identity, if $a \in Z_{sa}$ then $\phi(af) = a\phi(f)$. Thus by the above characterization of the center, the above isomorphism of M^α on M^β is the identity on Z_{sa}. By Theorem 3.3 $\alpha \sim \beta$.

Next assume M is of type II_∞. Since we may as above consider f and $\phi(f)$ for a subprojection f of e, we may assume $1 - e$ is infinite. We now divide e into four equivalent orthogonal projections, $e_1,\ldots e_4$ in M^α and $f_1,\ldots,f_4$ in M^β respectively, and find as before $M_e^\alpha \cong H_4(R_\alpha) \cong H_4(R_\beta) \cong M_e^\beta$ for some real $*$-algebras R_α and R_β. If we extend the projections $e_1,\ldots e_4$ to an orthogonal family of equivalent projections $(e_i)_{i\in J}$ in M^α with sum 1, we can as in the proof of [8, 7.6.3] find a copy N of $B(H)^{t_\infty}_{sa}$ in M^α with H a Hilbert space of dimension card J, such that the projections e_i are all minimal in N. From the action of the symmetries in N exchanging the e_i's and the fact that $M_e^\alpha = H_4(R_\alpha)$ it is easy to see that M^α is the JW-algebra generated by M_e^α and N. Since the similar result holds for M^β, it follows that $M^\alpha \cong M^\beta$ via a central isomorphism. Thus $\alpha \sim \beta$ by Theorem 3.3.

Finally assume M is of type III. Considering a subprojection of e in M^α if necessary we may assume $e \prec 1 - e$. But then the proof goes as in the II_∞-case. Q.E.D.

<u>REMARK</u>. One might expect that the converse to the above theorem to hold also. If $\alpha \sim \beta$ and ϕ is the central automorphism such that $\beta = \phi\alpha\phi^{-1}$ then ϕ is an isomorphism of M^α onto M^β such that $\phi(M_e^\alpha) = (M^\beta)_{\phi(e)}$. If M is finite, $\tau(e) = \tau(\phi(e))$ for all traces τ of M, hence $e \sim \phi(e)$ in M^β by the Comparison theorem [8, 5.2.13]. Thus $(M^\beta)_{\phi(e)} \cong (M^\beta)_e$, so by Theorem 3.3 $\alpha|M_e \sim \beta|M_e$. Similarly the same is true in the type I case.

In the type II_∞ case we cannot expect $e \sim \phi(e)$, hence that $(M^\beta)_{\phi(e)} \cong (M^\beta)_e$. Therefore it is probably false that $\alpha|M_e \sim \beta|M_e$.

In the type III case the converse holds if M has separable predual. This is a consequence of Theorem 3.3 and the following result.

LEMMA 3.6. Let M be a von Neumann algebra of type III with separable predual. Suppose α is a central involution on M and e a projection M^α with central support 1. Then $M^\alpha \cong (M^\alpha)_e$.

Proof. Consider $N = M \otimes B(H_2)$ with the central involution $\beta = \alpha \otimes t_2$, and let e_{ij}, $1 \leq i,j \leq 2$, be the matrix units in $B(H_2)$. Let p and q be orthogonal projections in N^β with the same central supports. Since N is of type III with separable predual there is $v \in N$ such that $v^* v = p$, $vv^* = q$. Let $w = v\alpha(v)$. Then $w^* w = p$, $ww^* = q$, and $\alpha(w) = w$. Let $s = w + w^*$. Then $s \in N^\beta$ and $sps = q$. Apply this to $e \otimes e_{11}$, $e \otimes e_{22}$ and $1 \otimes e_{22}$. Then $e \otimes e_{11} \sim e \otimes e_{22} \sim 1 \otimes e_{11}$ in N^β, so $e \otimes e_{11} \sim 1 \otimes e_{11}$ in N^β. It follows that $(N^\beta)_{e\otimes e_{11}} \cong (N^\beta)_{1\otimes e_{11}}$. Since $N_{1\otimes e_{11}} = \{a\otimes e_{11}: a \in M\}$, $(N^\beta)_{1\otimes e_{11}} = M^\alpha \otimes e_{11}$, and similarly $(N^\beta)_{e\otimes e_{11}} = (M^\alpha)_e \otimes e_{11}$. Thus $M^\alpha \cong (M^\alpha)_e$. Q.E.D.

4. Close involutions.

We show that involutions which are in a sense close, tend to be conjugate. Such results have previously been shown by Giordano, who showed two such results. If α is an involution on a von Neumann algebra M and u a unitary operator in M such that

$\alpha(u) = u$, then $\alpha\,\mathrm{Ad}u$ is an involution conjugate to α [5, Prop. 1.2], indeed they are conjugate via an inner automorphism. A deeper result is [5, Thm.1], which says that if M is a II_1-factor isomorphic to $M \circ R$, R the hyperfinite II_1-factor, then two involutions α and β are conjugate if $\alpha\beta \in \overline{\mathrm{Int}}\, M$ -the closure of the inner automorphisms $\mathrm{Int}\, M$. We shall now apply the first of these results to study involutions α and β such that $\alpha\beta \in \mathrm{Int}\, M$.

Recall that t_2 and q are the involutions of the complex 2×2 matrices $M_2(\mathbb{C})$ $(=B(H_2))$ defined by

$$t_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},\quad q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

PROPOSITION 4.1. Let M be a von Neumann algebra with two central involutions α and β such that their product $\alpha\beta$ is an inner automorphism of M. Then there are two central projections e and f in M with sum 1 such that $\alpha|Me \sim \beta|Me$, and $(\alpha|Mf) \otimes t_2 \sim (\beta|Mf) \otimes q$ (as involutions of $Mf \otimes M_2(C)$.

Proof. Let u be a unitary operator in M such that $\alpha\beta = \mathrm{Ad}u$. Let by Lemma 2.1 e and f be central projections in M with sum 1 such that $\alpha(u) = \beta(u) = (e-f)u$. Then $\alpha(eu) = \beta(eu) = eu$, so that $\alpha|Me \sim \beta|Me$ by quoted result of Giordano [5, Prop.1.2].

Next consider α and β restricted to Mf. We have $\alpha(fu) = \beta(fu) = -fu$. Furthermore, on $Mf \otimes M_2(\mathbb{C})$ we have $[(\alpha|Mf)\otimes t_2] \otimes [(\beta|Mf)\otimes q] = \mathrm{Ad}[fu\otimes\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}]$. Since

$$\alpha \otimes t_2(uf \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) = \beta \otimes q\,(uf\otimes\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) = uf \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

the first part of the proposition is applicable, hence $(\alpha|Mf) \otimes t_2 \sim (\beta|Mf) \otimes q$ on $Mf \otimes M_2(\mathbb{C})$. Q.E.D.

THEOREM 4.2. Let M be a von Neumann algebra. Suppose α and β are central involutions on M such that $\|\alpha-\beta\| < 2$. Then $\alpha \sim \beta$.

Proof. Since $\|\alpha\beta-\iota\| < 2$, $\alpha\beta$ is an inner automorphism Adu, [10]. Let e and f be central projections as in Proposition 4.1. The proof is complete if we can show $f = 0$. Assume $f \neq 0$ and consider Mf instead of M. We may thus assume $\alpha(u) = \beta(u) = -u$. Let $0 < \varepsilon < 1$, and let p be a spectral projection for u such that $\|pu-\lambda p\| < \varepsilon$, $|\lambda| = 1$. Then $\alpha(p)$ is a spectral projection for u such that $\|\alpha(p)u+\lambda\alpha(p)\| < \varepsilon$. In particular, since $p\alpha(p) = \alpha(p)p$, we have

$$\begin{aligned}\|\alpha(p)p\| &= \|\alpha(p)pu\| \\ &\leq \tfrac{1}{2}(\|\alpha(p)pu - \lambda\alpha(p)p\| + \|p\alpha(p)u + \lambda p\alpha(p)\|) \\ &\leq \tfrac{1}{2}\|pu-\lambda p\| + \tfrac{1}{2}\|\alpha(p)u+\lambda\alpha(p)\| \\ &< \varepsilon < 1.\end{aligned}$$

Thus $\alpha(p)p = 0$. Now it is easy to show $p \sim \alpha(p)$ as projections in M, see e.g. [12,Lem.3.3]. Say s is a self-adjoint operator in M such that $s^2 = p + \alpha(p)$ and $sps = \alpha(p)$. Since $\|pu^*-\bar{\lambda}p\| < \varepsilon$ and $\|\alpha(p)u^*+\bar{\lambda}\alpha(p)\| < \varepsilon$, we have

$$\begin{aligned}\|usu^*-s\| &= \|u(p+\alpha(p))su^*-s\| \\ &= \|u(ps+\alpha(p)s)u^*-s\| \\ &\geq \|(\lambda ps-\lambda\alpha(p)s)u^*-s\| - 2\varepsilon \\ &= \|(\lambda s\alpha(p)-\lambda sp)u^*-s\| - 2\varepsilon \\ &\geq \|\lambda s(-\bar{\lambda}\alpha(p)-\bar{\lambda}p)-s\| - 4\varepsilon \\ &= \|2s\| - 4\varepsilon = 2 - 4\varepsilon.\end{aligned}$$

Since ε is arbitrary, $\|\mathrm{Ad}u-\iota\| = 2$, contradicting the assumption $\|\alpha\beta-\iota\| < 2$. Therefore $f = 0$, completing the proof. Q.E.D.

5. Examples.

We exhibit factors of different types with one or more conjugacy classes of involutions. The type I case is described in Theorem 2.6, so we concentrate on types II and III.

Suppose first that M is a hyperfinite factor. If M is of type II_1 there is only one conjugacy class, see [6] or [14]. If M is of type III_λ, $0 < \lambda < 1$, there are exactly two conjugacy classes [4,Thm.6.4], see also [11]. If M is of type III_0, M can have 2^n, $n \in \mathbb{N} \cup \{\infty\}$, conjugacy classes [4,Prop.6.6.7]. The III_1-case in open except the ITPFI-example, in which case there is one conjugacy class [4,Thm.6.3]. The above conjugacy classes where distinguished by using automorphisms of order 2 of the "flow of weights".

To study the II_1-case we let G be a countable ICC-group and L its left regular representation. Then the inverse operation on G defines an involution θ on the II_1-factor $L(G)$ generated by L. The real $*$-algebra $R^\theta = \{x \in L(G) : \theta(x) = x^*\}$ is then the weakly closed real $*$-algebra generated by the image of L.

Let $G = \mathbb{F}_2$ -the free group in two generators a and b. Then $L(\mathbb{F}^2)$ has an automorphism γ given by $L(a) \to -L(a)$, $L(b) \to -L(b)$. The composition $\beta = \gamma\theta$ is another involution of $L(\mathbb{F}_2)$, and $\theta\beta = \gamma$ is an outer automorphism [2]. Still it follows from Corollary 3.4 that $\theta \sim \beta$. Indeed R^β is the real $*$-algebra generated by $iL(a)$ and $iL(b)$, which is isomorphic to R^θ via the map $iL(a) \to L(a)$, $iL(b) \to L(b)$.

If M is a factor with separable predual we let ε denote the canonical homomorphism $\varepsilon\colon \mathrm{Aut}\, M \to \mathrm{Out}\, M = \mathrm{Aut}\, M/\mathrm{Int}\, M$. Following [3] we let $\chi(M)$ denote the center of $\varepsilon(\overline{\mathrm{Int}}\, M)$. Since $\overline{\mathrm{Int}}\, M$ is a normal subgroup of $\mathrm{Aut}\, M$ so is $\chi(M)$ of $\mathrm{Out}\, M$. Following ideas of Jones [9] if α is an automorphism or anti-automorphism of M we denote by $\tilde{\alpha}$ the automorphism of $\mathrm{Out}\, M$ and $\chi(M)$ given by $\tilde{\alpha}(\varepsilon(\gamma)) = \varepsilon(\alpha\gamma\alpha^{-1})$. Then the map $\alpha \to \tilde{\alpha}$ is a homomorphism. We let

$$\chi(M)^{\alpha} = \{\varepsilon(\gamma) \in \chi(M) : \tilde{\alpha}(\varepsilon(\gamma)) = \varepsilon(\gamma)\}$$

be the fixed point group of $\tilde{\alpha}$ in $\chi(M)$.

LEMMA 5.1. Let M be a factor with separable predual. Suppose α and β are involutions on M. If $\alpha \sim \beta$ then $\chi(M)^{\alpha} \cong \chi(M)^{\beta}$.

Proof. Suppose $\phi \in \mathrm{Aut}\, M$ and $\beta = \phi\alpha\phi^{-1}$. Then $\tilde{\phi}$ is the desired isomorphism. Indeed, $\tilde{\phi}$ restricts to an automorphism of $\chi(M)$, and if $\varepsilon(\gamma) \in \chi(M)^{\alpha}$ then

$$\tilde{\beta}(\tilde{\phi}(\varepsilon(\gamma))) = \tilde{\phi}\,\tilde{\alpha}\,\tilde{\phi}^{-1}(\tilde{\phi}(\varepsilon(\gamma))) = \tilde{\phi}(\varepsilon(\gamma)),$$

proving that $\tilde{\phi}(\varepsilon(\gamma)) \in \chi(M)^{\beta}$. Similarly $\tilde{\phi}^{-1}\colon \chi(M)^{\beta} \to \chi(M)^{\alpha}$.

PROPOSITION 5.2. For each $n \in \mathbb{N}$ there is a II_1-factor with separable predual having at least n distinct conjugacy classes of involutions.

Proof. Our examples will be factors of the form $M \otimes M \otimes \dots \otimes M$ as constructed by Connes in [3]. Let θ be the involution on $L(\mathbb{F}_2)$ defined by the inverse operation, and let $\overset{\infty}{\underset{1}{\otimes}}\, \theta$ be the infinite tensor product of θ with itself on $N = \overset{\infty}{\underset{1}{\otimes}}\, L(\mathbb{F}_2)$. Let β' be the automorphism of $L(\mathbb{F}_2)$ obtained by flipping the two generators, and let β be the infinite tensor product of β' with itself on N. With G the group $\{1,\beta\}$ and M the crossed product of N and G, $\overset{\infty}{\underset{1}{\otimes}}\theta$ extends to an involution α on M as in [9, Lem. 4.1]. We let M^{2n} be the tensor product of M with itself 2n times written in the form

$M^{2n} = (M\otimes M) \otimes (M\otimes M) \otimes \ldots \otimes (M\otimes M)$.

Let α' be the involution $\alpha \otimes \alpha$ on $M \otimes M$, and σ' the involution $\alpha'\sigma$ on $M \otimes M$, where σ is the Sakai flip on $M \otimes M$, $\sigma(x\otimes y) = y \otimes x$. Let α_k, $k = 0,1,\ldots,n$, be involutions on M^{2n} defined by

$$\alpha_k = \underbrace{\sigma' \otimes \ldots \otimes \sigma'}_{k\text{ times}} \otimes \underbrace{\alpha' \otimes \ldots \otimes \alpha'}_{n-k\text{ times}}.$$

By [3] $\chi(M) = Z_2$, and $\chi(M^{2n}) = Z_2^{2n}$. Since the only automorphism of Z_2 is the identity, $\tilde{\alpha}$ is the identity map, hence so is $\tilde{\alpha}'$. If $\gamma_1, \gamma_2 \in \operatorname{Aut} M$ then $\sigma(\gamma_1\otimes\gamma_2)\sigma = \gamma_2 \otimes \gamma_1$, hence $\tilde{\sigma}'$ is the flip on $\chi(M\otimes M) = \mathbb{Z}_2\times \mathbb{Z}_2$. It follows that

$$\chi(M\otimes M)^{\tilde{\sigma}} = \{(0,0),(1,1)\} \cong Z_2.$$

We therefore have that

$$\chi(M^{2n})^{\alpha_k} = \underbrace{\mathbb{Z}_2\times\ldots\times \mathbb{Z}_2}_{k\text{ times}} \times \underbrace{(\mathbb{Z}_2\times \mathbb{Z}_2)\times\ldots\times(\mathbb{Z}_2\times \mathbb{Z}_2)}_{n-k\text{ times}}$$

$$= \mathbb{Z}_2^{2n-k}$$

By Lemma 5.1 $\alpha_0,\ldots,\alpha_n$ are mutually non-conjugate. Q.E.D

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